Linearizable Initial-Boundary Value Problems for the sine-Gordon Equation on the Half-Line

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Dedicated to I.M. Gelfand
on the occasion of his ninetieth birthday

September 2003

Abstract

A rigorous methodology for the analysis of initial boundary value problems on the half-line, $0 < x < \infty$, $t > 0$, for integrable nonlinear evolution PDEs has recently appeared in the literature. As an application of this methodology the solution $q(x,t)$ of the sine-Gordon equation can be obtained in terms of the solution of a 2×2 matrix Riemann-Hilbert problem. This problem is formulated in the complex k-plane and is uniquely defined in terms of the so called spectral functions $a(k)$, $b(k)$, and $B(k)/A(k)$. The functions $a(k)$ and $b(k)$ can be constructed in terms of the given initial conditions $q(x,0)$ and $q_t(x,0)$ via the solution of a system of two *linear* ODE's, while for *arbitrary* boundary conditions the functions $A(k)$ and $B(k)$ can be constructed in terms of the given boundary condition via the solution of a system of four *nonlinear* ODEs. In this paper we analyse two *particular* boundary conditions: the case of constant Dirichlet data, $q(0,t) = \chi$, as well as the case that $q_x(0,t)$, $\sin(q(0,t)/2)$, and $\cos(q(0,t)/2)$ are linearly related by two constants χ_1 and χ_2. We show that for these particular cases, the system of the above nonlinear ODEs can be avoided, and $B(k)/A(k)$ can be computed explicitly in terms of $\{a(k), b(k), \chi\}$ and $\{a(k), b(k), \chi_1, \chi_2\}$ respectively. Thus these "linearizable" initial-boundary value problems can be solved with absolutely the same level of efficiency as the classical initial value problem of the line.

1 Introduction

Let the real function $q(x,t)$ satisfy an initial-boundary value problem for the sine-Gordon equation on the half-line, $\{0 < x < \infty, 0 < t < T\}$, where T is a positive constant. The

function $q(x,t)$ can be constructed as follows [1],[2]:

- Given initial conditions construct the spectral functions $a(k)$ and $b(k)$. These functions are defined in terms of $\phi(x,k)$, where the vector ϕ is an appropriate solution of the x-part of the associated Lax pair evaluated at $t=0$. Thus ϕ is defined in terms of the initial conditions $q(x,0)$ and $q_t(x,0)$.

- From the given boundary condition, characterize the unknown boundary value at $x=0$ by the requirement that the spectral functions $\{a(k), b(k), A(k), B(k)\}$ satisfy the global relation

$$a(k)B(k) - b(k)A(k) = e^{\frac{i}{4}(k+\frac{1}{k})T}c(k), \quad \operatorname{Im} k \geq 0, \quad k \neq 0, \tag{1.1}$$

  where $c(k)$ is analytic for $\operatorname{Im} k > 0$ and is of $O(1/k)$ as $k \to \infty$. The functions $A(k)$ and $B(k)$ are defined in terms of $\Phi(t,k)$, where the vector Φ is an appropriate solution of the t-part of the associated Lax pair evaluated at $x=0$. Thus Φ is defined in terms of $q(0,t)$ and $q_x(0,t)$.

- Given $\{a(k), b(k), B(k)/A(k)\}$, construct $q(x,t)$ through the solution of a 2×2 matrix Riemann-Hilbert problem. The function $q(x,t)$ satisfies the sine-Gordon equation as well as the given initial and boundary conditions.

The most complicated step in the above construction is the characterization of the missing boundary value. For example, for the Dirichlet problem where the function $q(0,t)$ is prescribed as the boundary condition, it is shown in [2] that the unknown boundary value $q_x(0,t)$ can be obtained through the solution of a system of four *nonlinear* ODEs.

It was shown in [1] and [3] that for some particular boundary conditions, which we refer to as *linearizable* boundary conditions, it is possible to bypass the above system of nonlinear ODEs and to construct $B(k)/A(k)$ using only algebraic manipulations. In particular, it was shown in [1] that this is the case for the boundary condition $q(0,t) = \chi$, χ constant.

In this paper we show that there exists another linearizable boundary condition which involves two constants χ_1 and χ_2. For completeness we also include the case $q(0,t) = \chi$.

Theorem 1.1 Let the real function $q(x,t)$ satisfy the sine-Gordon equation

$$q_{tt} - q_{xx} + \sin q = 0, \quad 0 < x < \infty, \quad 0 < t < T, \tag{1.2}$$

(where T is a given constant) the initial conditions

$$q(x,0) = q_0(x), \quad q_t(x,0) = q_1(x), \quad 0 < x < \infty, \tag{1.3}$$

and either of the following two boundary conditions,

$$q(0,t) = \chi, \tag{1.4a}$$

or

$$q_x(0,t) + \chi_1 \cos\left(\frac{q(0,t)}{2}\right) + \chi_2 \sin\left(\frac{q(0,t)}{2}\right) = 0, \tag{1.4b}$$

where χ, χ_1, χ_2 are real constants. Assume that $q_0(x)-2\pi m$ and $q_1(x)$ are Schwartz functions for m integer, and that the initial conditions are compatible with the boundary conditions at $x=t=0$, i.e. assume that for (1.4a) and (1.4b) the following conditions are valid respectively

$$q_0(0)=\chi \quad \text{and} \quad q_1(0)=0, \qquad \dot{q}_0(0)+\chi_1\cos\left(\frac{q_0(0)}{2}\right)+\chi_2\sin\left(\frac{q_0(0)}{2}\right)=0.$$

The above initial-boundary value problems have a unique global solution given by

$$\cos q(x,t)=1+2\lim_{k\to\infty}\left\{(k\mu_{12}(x,t,k))^2+2i\partial_x(k\mu_{22}(x,t,k))\right\}, \tag{1.5}$$

where μ_{12} and μ_{22} denote the (12) and (22) entries of the 2×2 matrix $\mu(x,t,k)$ which satisfies the following Riemann-Hilbert problem.

(i) μ is meromorphic in k for $k\in\mathbb{C}\backslash\mathcal{L}$, where $\mathcal{L}$, which is depicted in Figure 1, is defined by

$$\mathcal{L}=\{\text{Re } k=0\cup|k|=1\}.$$

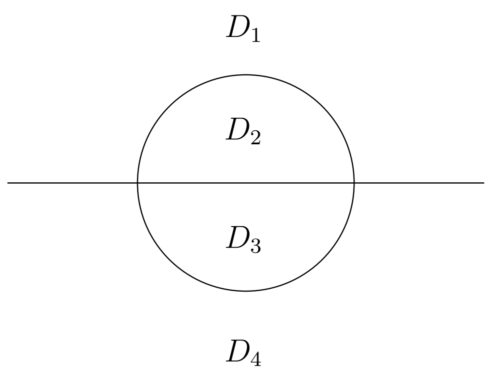

Figure 1: The contour $\mathcal{L}$ and the domains $D_j, j=1,\cdots,4$.

(ii) Let the domains $D_j, j=1,\cdots,4$ which are depicted in Figure 1, be defined by

$$D_1=\{\text{Im } \ k>0\cap|k|>1\}, \quad D_2=\{\text{Im } k>0\cap|k|<1\},$$

$$D_3=\{\text{Im } \ k<0\cap|k|<1\}, \quad D_4=\{\text{Im } k<0\cap|k|>1\}.$$

The matrix μ satisfies the jump condition

$$\mu_-(x,t,k)=\mu_+(x,t,k)J(x,t,k), \quad k\in\mathcal{L}, \tag{1.6}$$

where μ is μ_- for $k\in D_2\cup D_4$, μ is μ_+ for $k\in D_1\cup D_3$, and the 2×2 matrix J is defined in terms of the spectral functions $\{a(k),b(k),B(k)/A(k)\}$ and the explicit function $\theta(x,t,k)$ by the following formulae:

$$J=J_1, \quad k\in D_1\cap D_2; \qquad J=J_2, \quad k\in D_2\cap D_3;$$

$$J=J_3, \quad k\in D_3\cap D_4; \qquad J=J_4, \quad k\in D_4\cap D_1,$$

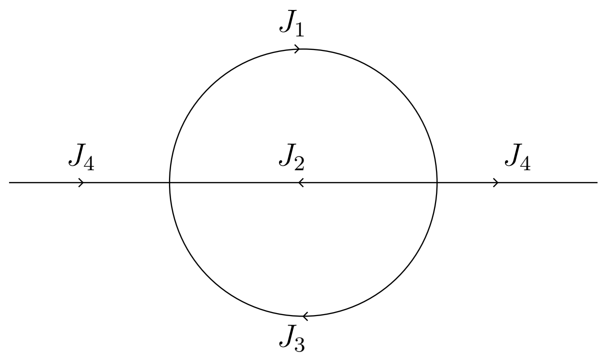

Figure 2: The jump matrix J

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\theta} & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & \overline{\Gamma(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ -\bar{\gamma}(k)e^{2i\theta} & 1+|\gamma(k)|^2 \end{pmatrix}, \quad J_2 = J_3 J_4^{-1} J_1,$$

$$\gamma(k) = \frac{b(k)}{\overline{a(k)}}, k \in \mathbb{R}; \quad \Gamma(k) = -\frac{\overline{\frac{B(\bar{k})}{A(\bar{k})}}}{a(k)\left[a(k) + b(k)\overline{\frac{B(\bar{k})}{A(\bar{k})}}\right]}, \quad k \in D_2; \tag{1.7}$$

$$\theta(x,t,k) = \frac{1}{4}(k - \frac{1}{k})x + \frac{1}{4}(k + \frac{1}{k})t.$$

The functions $a(k)$ and $b(k)$ are defined in terms of $q_0(x)$ and $q_1(x)$ by

$$a(k) = \phi_2(0,k), \quad b(k) = \phi_1(0,k), \quad \text{Im} \ \ k \geq 0, \tag{1.8}$$

where the vector $\phi(x,k)$ with component $\phi_1(x,k)$ and $\phi_2(x,k)$ satisfies

$$\phi_x + \frac{i}{4}\left(k - \frac{1}{k}\right)\sigma_3\phi = \frac{1}{4}\begin{pmatrix} i(k + \frac{\cos q_0(x)}{k}) & -i(\dot{q}_0(x) + q_1(x)) - \frac{\sin q_0(x)}{k} \\ -i(\dot{q}_0(x) + q_1(x)) + \frac{\sin q_0(x)}{k} & -i(k + \frac{\cos q_0(x)}{k}) \end{pmatrix}\phi,$$

$$0 < x < \infty, \quad \text{Im} \ \ k \geq 0,$$

$$\phi(x,k) = e^{ikx}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right) \quad \text{as} \quad x \to \infty, \tag{1.9}$$

and $\sigma_3 = \text{diag}\ (1, -1)$.

For $k \in D_1$ the ratio $B(k)/A(k)$ equals $b(k)/a(k)$. For $k \in D_3$, the ratio $B(k)/A(k)$ for the cases (1.4a) and (1.4b) is given respectively by

$$\frac{B(k)}{A(k)} = \frac{f(k)b(\frac{1}{k}) - a(\frac{1}{k})}{f(k)a(\frac{1}{k}) - b(\frac{1}{k})}, \tag{1.10a}$$

and

$$\frac{B(k)}{A(k)} = \frac{b(\frac{1}{k})[\alpha(k)+\overline{\alpha(\bar{k})}+2i\cos(\frac{q_0(0)}{2})]+a(\frac{1}{k})[\overline{\alpha(\bar{k})}-\alpha(k)-2\sin(\frac{q_0(0)}{2})]}{a(\frac{1}{k})[\alpha(k)+\overline{\alpha(\bar{k})}-2i\cos(\frac{q_0(0)}{2})]-b(\frac{1}{k})[\alpha(k)-\overline{\alpha(\bar{k})}-2\sin(\frac{q_0(0)}{2})]} \qquad (1.10b)$$

where

$$f(k) = i\frac{k^2+1}{k^2-1}\frac{\sin\chi}{\cos\chi-1}, \quad \alpha(k) = \frac{i\chi_1}{k+\frac{1}{k}} + \frac{\chi_2}{k-\frac{1}{k}}. \qquad (1.11)$$

(iii) Define the function $\Delta(k)$ by

$$\Delta(k) = a(k)\overline{D(\bar{k})} + b(k)\overline{N(\bar{k})}, \qquad (1.12),$$

where $D(k)$ and $N(k)$ denote the denominator and the numerator of the rhs of equations (1.10). If the function $a(k)$ has zeros for Im $k > 0$, and/or the function $\Delta(k)$ has zeros in D_2, then $\mu(x,t,k)$ satisfies appropriate residue conditions, see section 4.

Organization of the Paper In section 2 we summarize the methodology of [1], [3] for identifying and analyzing linearizable boundary conditions. If a given PDE admits different Lax pair [4] formulations, it is possible to search for linearizable boundary conditions for each of these different formulations. The Lax pair analyzed in [1], see equations (2.9), gives rise to the case (1.4a); the sG also admits an alternative Lax pair [5], see (2.11), which gives rise to the linearizable case (1.4b). Since the basic RH problem presented in Theorem 1.1 is associated with the Lax pair analyzed in [1], we present in section 3 a general formalism which connects the spectral functions $\{A(k), B(k)\}$ associated with two different Lax pairs. As an application of this formalism we use the alternative Lax pair (2.11) to identify (1.4b), but we solve the sG, for both boundary conditions (1.4a) and (1.4b), using the Lax pair of [1]. In section 4 we derive the residue conditions and in section 5 we discuss further these results.

2 An Overview, Different Lax Pairs for the sine-Gordon, and Linearizable Boundary Conditions

We first recall the definition of the spectral functions $A(k)$ and $B(k)$ [1].

Definition 2.1 Suppose that the t-part of the Lax pair of a given integrable PDE is

$$\Psi_t + if_2(k)\sigma_3\Psi = \tilde{Q}(x,t,k)\Psi, \qquad (2.1)$$

where

$$\sigma_3 = \mathrm{diag}(1,-1), \qquad (2.2)$$

$\Psi(x,t,k)$ is a 2×2 matrix, the scalar $f_2(k)$ is an analytic function of k, and the 2×2 matrix $\tilde{Q}(x,t,k)$ is an analytic function of k, of $q(x,t)$, of $\bar{q}(x,t)$, and of the derivatives of these functions. Let $M(t,k)$ be the unique solution of

$$M_t + if_2(k)\sigma_3 M = \tilde{Q}(0,t,k)M, \quad 0 < t < T, \quad k \in \mathbb{C},$$

$$M(0,k) = \mathrm{diag}(1,1), \tag{2.3}$$

where T is a positive constant. Assume that $\tilde{Q}(0,t,k)$ is such that $M(t,k)$ has the form

$$M(t,k) = \begin{pmatrix} \overline{\Phi_2(t,\bar{k})} & \Phi_1(t,k) \\ \rho\overline{\Phi_1(t,\bar{k})} & \Phi_2(t,k) \end{pmatrix}, \quad \rho^2 = 1. \tag{2.4}$$

The spectral functions $A(k)$ and $B(k)$ are defined by

$$A(k) = e^{if_2(k)T}\overline{\Phi_2(T,\bar{k})}, \quad B(k) = -e^{if_2(k)T}\Phi_1(T,k), \quad k \in \mathbb{C}. \tag{2.5}$$

Linearizable Boundary Conditions

Let the transformation $k \to \nu(k)$ be defined by the requirement that it leaves $f_2(k)$ invariant, i.e.

$$f_2(\nu(k)) = f_2(k), \quad k \in \mathbb{C}. \tag{2.6}$$

Suppose that there exists a nonsingular 2×2 matrix $N(k)$ such that

$$N(k)^{-1}\left[if_2(k)\sigma_3 - \tilde{Q}(0,t,\nu(k))\right]N(k) = if_2(k)\sigma_3 - \tilde{Q}(0,t,k). \tag{2.7}$$

Then the solution $M(t,k)$ of equations (2.3) satisfies

$$M(t,\nu(k)) = N(k)M(t,k)N(k)^{-1}. \tag{2.8}$$

This equation and the definitions (2.5), imply a relation between $\{A(\nu(k)), B(\nu(k))\}$ and $\{A(k), B(k)\}$. Using this relation, it is possible to compute $B(k)/A(k)$ using only the algebraic manipulation of the global relation [1].

Two Different Lax Pairs for the sine-Gordon

The sG possesses the Lax pair [6]

$$\Psi_x + \frac{i}{4}(k - \frac{1}{k})\sigma_3\Psi = Q(x,t,k)\Psi,$$

$$\Psi_t + \frac{i}{4}(k + \frac{1}{k})\sigma_3\Psi = \tilde{Q}(x,t,k)\Psi, \quad k \in \mathbb{C}, \quad k \neq 0, \tag{2.9}$$

where $\Psi(x,t,k)$ is a 2×2 matrix, $\tilde{Q}(x,t,k) = Q(x,t,-k)$, and the 2×2 matrix $Q(x,t,k)$ is defined by

$$Q(x,t,k) = \frac{1}{4}\begin{pmatrix} \frac{i}{k}(-1+\cos q) & -i(q_x+q_t) - \frac{\sin q}{k} \\ -i(q_x+q_t) + \frac{\sin q}{k} & -\frac{i}{k}(-1+\cos q) \end{pmatrix} \tag{2.10}$$

with q denoting $q(x,t)$.

The sG also possesses the alternative Lax pair [5]

$$\Psi_x = \mathcal{U}(x,t,k)\Psi, \quad \Psi_t = \mathcal{V}(x,t,k)\Psi, \quad k \in \mathbb{C}, \quad k \neq 0, \tag{2.11}$$

where $\Psi(x,t,k)$ is a 2×2 matrix and the 2×2 matrices $\mathcal{U}(x,t,k)$ and $\mathcal{V}(x,t,k)$ are defined by

$$\mathcal{U}=\frac{1}{4}\begin{pmatrix} -iq_t & ke^{\frac{iq}{2}}-\frac{1}{k}e^{-\frac{iq}{2}} \\ -ke^{-\frac{iq}{2}}+\frac{1}{k}e^{\frac{iq}{2}} & iq_t \end{pmatrix}, \quad \mathcal{V}=\frac{1}{4}\begin{pmatrix} -iq_x & ke^{\frac{iq}{2}}+\frac{1}{k}e^{-\frac{iq}{2}} \\ -ke^{-\frac{iq}{2}}-\frac{1}{k}e^{\frac{iq}{2}} & iq_x \end{pmatrix}. \tag{2.12}$$

Linearizable Case of the Second Lax Pair We now show that by analyzing the t-part of the second Lax pair (2.11) evaluated at $x=0$, it is possible to identify linearizable boundary conditions using the simple formulation reviewed earlier, see equations (2.6)-(2.8).

Proposition 2.1 Let the 2×2 matrix $\mathcal{M}(t,k)$ be the following solution of the t-part of the Lax pair (2.11) evaluated at $x=0$:

$$\begin{gathered}\mathcal{M}_t=\mathcal{V}(t,k)\mathcal{M}, \quad 0<t<T, \quad k\in\mathbb{C}, \\ \mathcal{M}(0,k)=\mathrm{diag}(1,1),\end{gathered} \tag{2.13}$$

where $\mathcal{V}(t,k)=\mathcal{V}(0,t,k)$ and $\mathcal{V}(x,t,k)$ is defined by equation (2.12b). Then $\mathcal{M}$ satisfies the "symmetry" relation

$$\mathcal{M}(t,\frac{1}{k})=N(k)\mathcal{M}(t,k)N(k)^{-1}, \quad 0<t<T, \quad k\in\mathbb{C}, \quad k\neq 0, \tag{2.14}$$

where for equations (1.4a) and (1.4b), $N(k)$ is given respectively by

$$N(k)=\begin{pmatrix} 1 & 0 \\ 0 & \frac{e^{-\frac{i\chi}{2}}+k^2e^{\frac{i\chi}{2}}}{e^{\frac{i\chi}{2}}+k^2e^{-\frac{i\chi}{2}}} \end{pmatrix} \tag{2.15}$$

and

$$N(k)=\begin{pmatrix} \frac{i\chi_1}{k+\frac{1}{k}}+\frac{\chi_2}{k-\frac{1}{k}} & 1 \\ -1 & \frac{\chi_2}{k-\frac{1}{k}}-\frac{i\chi_1}{k+\frac{1}{k}} \end{pmatrix}. \tag{2.16}$$

Proof In this case $f_2(k)=(k+1/k)/4$, thus equation (2.6) implies $\nu=1/k$. Let us introduce the following notations for the matrix $N(k)$,

$$N_{11}=N_1, \quad N_{12}=N_2, \quad N_{21}=N_3, \quad N_{22}=N_4. \tag{2.17}$$

In this case equation (2.8) is

$$\mathcal{V}(t,\nu(k))=N(k)\mathcal{V}(t,k)N(k)^{-1}. \tag{2.18}$$

If $N_3=-N_2$, then the (11) and (22) entries of this equation are identically satisfied, while both the (12) and (21) entries of equation (2.18) yield the equation

$$2iq_x(0,t)N_2+e^{-\frac{iq}{2}(0,t)}\left(\frac{N_1}{k}-kN_4\right)+e^{\frac{iq}{2}(0,t)}\left(kN_1-\frac{N_4}{k}\right)=0. \tag{2.19}$$

We distinguish two cases

(i) $N_2 = 0, \quad N_4 = \alpha(k)N_1$.

If $\alpha(k)$ equals the (22) entry of the matrix N defined in (2.15) then equation (2.19) becomes the boundary condition (1.4a).

(ii) $N_2 \neq 0$, $N_1 = \alpha(k)N_2$, $N_4 = \beta(k)N_2$.

Then

$$q_x(0,t) + \frac{i}{2}(\beta-\alpha)(k+\frac{1}{k})\cos\left(\frac{q(0,t)}{2}\right) + \frac{1}{2}(\beta+\alpha)(k-\frac{1}{k})\sin\left(\frac{q(0,t)}{2}\right) = 0. \tag{2.20}$$

Letting

$$\frac{i}{2}(\beta-\alpha)(k+\frac{1}{k}) = \chi_1, \quad \frac{1}{2}(\beta+\alpha)(k-\frac{1}{k}) = \chi_2,$$

i.e.

$$\alpha(k) = \frac{i\chi_1}{k+\frac{1}{k}} + \frac{\chi_2}{k-\frac{1}{k}}, \quad \beta(k) = \frac{\chi_2}{k-\frac{1}{k}} - \frac{i\chi_1}{k+\frac{1}{k}} = \overline{\alpha(\bar{k})}, \tag{2.21}$$

equation (2.20) becomes the boundary condition (1.4b), and $N(k)$ is given by equation (2.16).

QED

3 Spectral Functions Associated with Different Lax Pairs

Equation (2.14) expressed a "symmetry" relation for the eigenfunction $\mathcal{M}(t,k)$ associated with the second Lax pair (2.11). The spectral functions $A(k)$ and $B(k)$ are defined in terms of the eigenfunction $M(t,k)$ associated with the first Lax pair (2.9). In what follows we present a general result which, starting with the "symmetry" relation satisfied by $\mathcal{M}(t,k)$, yields the "symmetry" relations satisfied by $\{A(k), B(k)\}$.

Proposition 3.1 Let $0 < t < T$, $k \in \mathbb{C}$, $k \neq 0$. Let $M(t,k)$ be the unique solution of the 2×2 matrix equations

$$M_t = V(t,k)M, \quad V(t,k) = \tilde{Q}(0,t,k) - if_2(k)\sigma_3,$$

$$M(0,k) = \mathrm{diag}(1,1), \tag{3.1}$$

where $\tilde{Q}, f_2, \sigma_3$ are as in Definition 2.1 (see equation (2.1)). Assume that $M(t,k)$ has the form (2.4). Define the spectral functions $A(k)$ and $B(k)$ by equations (2.5) where $f_2(k)$ satisfies the relation (2.6).

Let $\mathcal{M}(t,k)$ be the unique solution of the 2×2 matrix equations

$$\mathcal{M}_t = \mathcal{V}(t,k)\mathcal{M},$$

$$\mathcal{M}(0,k) = \mathrm{diag}(1,1). \tag{3.2}$$

Suppose that there exists a non-singular 2×2 matrix $H(t)$ such that

$$H_t(t) = \mathcal{V}(t,k)H(t) - H(t)V(t,k). \tag{3.3}$$

Assume that $\mathcal{M}(t,k)$ satisfies the "symmetry" relation

$$\mathcal{M}(t,\nu(k)) = N(k)\mathcal{M}(t,k)N(k)^{-1}, \tag{3.4}$$

where $N(k)$ is a 2×2 nonsingular matrix.

Then $M(t,k)$ satisfies the symmetry relation

$$M(t,\nu(k)) = F(t,k)M(t,k)F(0,k)^{-1}, \quad F(t,k) = H(t)^{-1}N(k)H(t). \tag{3.5}$$

Furthermore, the spectral functions $A(k)$ and $B(k)$ satisfy the symmetry relation

$$\begin{pmatrix} A(\nu(k))e^{-if_2(k)T} & -B(\nu(k))e^{-if_2(k)T} \\ -\rho\overline{B(\nu(\bar{k})}e^{\overline{if_2(\bar{k})T}} & \overline{A(\nu(\bar{k})}e^{\overline{if_2(\bar{k})T}} \end{pmatrix} =$$

$$H(T)^{-1}N(k)H(T)\begin{pmatrix} A(k)e^{-if_2(k)T} & -B(k)e^{-if_2(k)T} \\ -\rho\overline{B(\bar{k})}e^{\overline{if_2(\bar{k})T}} & \overline{A(\bar{k})}e^{\overline{if_2(\bar{k})T}} \end{pmatrix} H(0)^{-1}N(k)^{-1}H(0). \tag{3.6}$$

Proof. If the 2×2 matrices $\mathcal{V}(t,k)$ and $V(t,k)$ are related by equation (3.3), then the 2×2 matrices $\mathcal{M}(t,k)$ and $M(t,k)$ are related by the equation

$$\mathcal{M}(t,k) = H(t)M(t,k)H(0)^{-1}. \tag{3.7}$$

Indeed, equation (3.7) is identically satisfied at $t=0$. Furthermore replacing in equation (3.2a) $\mathcal{M}(t,k)$ by the rhs of equation (3.7) we find

$$H_t M H(0)^{-1} + H M_t H(0)^{-1} = \mathcal{V} H M H(0)^{-1}.$$

Replacing in this equation M_t by VM, and using equation (3.3) we find an identity.

Since $\mathcal{M}(t,k)$ satisfies the "symmetry" relation (3.4), and $M(t,k)$ is related with $\mathcal{M}(t,k)$ through equation (3.7), it is easy to show that $M(t,k)$ satisfies the symmetry relation (3.5). Indeed, this latter equation follows from (3.7) by replacing k with $\nu(k)$ and then using equations (3.4) and (3.7).

Having established the "symmetry" relation (3.5) it is straightforward to obtain a symmetry relation for the spectral functions $\{A(k), B(k)\}$. Indeed, evaluating equation (3.5) at $t=T$, and expressing $M(T,k)$ in terms of the spectral functions (see equations (2.4), (2.5)), equation (3.5) yields equation (3.6). **QED**

The sG Case For the sG equation

$$f_2(k) = \frac{1}{4}(k+\frac{1}{k}),$$

$$V(t,k) = Q(0,t,-k) - \frac{i}{4}(k+\frac{1}{k})\sigma_3, \quad \mathcal{V}(t,k) = \mathcal{V}(0,t,k), \tag{3.8}$$

where $Q(x,t,k)$ and $\mathcal{V}(x,t,k)$ are defined by equations (2.10) and (2.12b) respectively. It can be verified that in this case $M(t,k)$ has the form (2.4) with $\rho=-1$. Furthermore, equation (3.3) is valid with the 2×2 nonsingular matrix $H(t)$ given by

$$H(t)=e^{-\frac{i\pi}{4}}\begin{pmatrix} e^{\frac{iq(0,t)}{4}} & e^{\frac{iq(0,t)}{4}} \\ -ie^{\frac{-iq(0,t)}{4}} & ie^{\frac{-iq(0,t)}{4}} \end{pmatrix}. \tag{3.9}$$

Thus for the boundary condition (1.4b), the spectral functions $A(k)$ and $B(k)$ satisfy equation (3.6) with $f_2(k)$, $H(t)$, and $N(k)$ given by equations (3.8a), (3.9), and (2.16) respectively.

Proof of Theorem 1.1 Let $q(x,t)$ be defined by equation (1.5) in terms of the solution $\mu(x,t,k)$ of the 2×2 matrix RH problem with the jump condition (1.6), where the jump matrix J is defined by equations (1.7) in terms of $\{a(k),b(k),A(k),B(k)\}$. Let $a(k)$ and $b(k)$ be defined by equations (1.8) and (1.9). Let $A(k)$ and $B(k)$ be defined by equations (2.3)–(2.5) where $\tilde{Q}(0,t,k)$ in equation (2.3) equals $Q(0,t,-k)$ which is defined by equation (2.10) with $q(0,t)$ and $q_x(0,t)$ replaced by $g_0(t)$ and $g_1(t)$. It was shown in [1] that if $g_0(t)$ and $g_1(t)$ are such that the global relation (1.1) is valid, then $q(x,t)$ satisfies the sG, and also $q(x,0)=q_0(x)$, $q_t(x,0)=q_1(t)$, $q(0,t)=g_0(t)$, $q_x(0,t)=g_1(t)$.

Furthermore, it was shown in [1] that if $g_0(t)=\chi$ then the definition of $\{A(k),B(k)\}$ and the global relation (1.1) imply (1.10a). Thus it only remains to be shown that if the boundary condition (1.4b) is valid then B/A satisfies equation (1.10b). In this respect we note that the definition of $\{A(k),B(k)\}$ implies the symmetry relation (3.6). In what follows we show that this relation together with the global relation (1.1) imply (1.10b). For this purpose it is convenient to assume that $q(x,t)\to 0$ as $t\to\infty$ and to let $T\to\infty$. In this case $A(k)$ and $B(k)$ are *not* entire functions but are analytic functions for $k\in D_1\cup D_3$. Also, the global relation (1.1) becomes

$$a(k)B(k)-b(k)A(k)=0, \quad k\in D_1. \tag{3.10}$$

Note that $\{A(k),B(k)\}$ are defined for $k\in D_1\cup D_3$, while $\{a(k),b(k)\}$ are defined for $k\in D_1\cup D_2$, thus each term of equation (3.10) is well defined for $k\in D_1$.

The assumption $q(0,t)\to 0$ as $t\to\infty$, and the definitions of $H(t)$ and $N(k)$, ie. equations (3.9) and (2.16), imply

$$H(\infty)^{-1}N(k)H(\infty)=\frac{1}{2}\begin{pmatrix} \alpha+\bar{\alpha}-2i & \alpha-\bar{\alpha} \\ \alpha-\bar{\alpha} & \alpha+\bar{\alpha}+2i \end{pmatrix}, \tag{3.11}$$

where $\alpha(k)$ is defined by equation (2.21a). Similarly

$$H(0)^{-1}N(k)^{-1}H(0)=\frac{1}{2(1+\alpha\bar{\alpha})}\begin{pmatrix} \alpha+\bar{\alpha}+2i\cos\left(\frac{q_0(0)}{2}\right) & \bar{\alpha}-\alpha-2\sin\left(\frac{q_0(0)}{2}\right) \\ \bar{\alpha}-\alpha+2\sin\left(\frac{q_0(0)}{2}\right) & \alpha+\bar{\alpha}-2i\cos\left(\frac{q_0(0)}{2}\right) \end{pmatrix}. \tag{3.12}$$

Using equations (3.11) and (3.12) in equation (3.6) (with $\rho=-1$), solving for $A(1/k)$, $B(1/k)$,

and letting $T \to \infty$, we find

$$\begin{aligned} A(\frac{1}{k}) &= \frac{\alpha+\bar{\alpha}-2i}{4(1+\alpha\bar{\alpha})}\Big\{[\alpha+\bar{\alpha}+2i\cos{(\frac{q_0(0)}{2})}]A(k)-[\bar{\alpha}-\alpha+2\sin{(\frac{q_0(0)}{2})}]B(k)\Big\}, \\ B(\frac{1}{k}) &= \frac{\alpha+\bar{\alpha}-2i}{4(1+\alpha\bar{\alpha})}\Big\{[\alpha+\bar{\alpha}-2i\cos{(\frac{q_0(0)}{2})}]B(k)-[\bar{\alpha}-\alpha-2\sin{(\frac{q_0(0)}{2})}]A(k)\Big\}, \end{aligned} \tag{3.13}$$

where $k \in D_1 \cup D_3$.

The "symmetry" relations (3.13) together with the global relation (3.10), yield $B(k)/A(k)$ in terms of $\{a(k), b(k)\}$. Indeed, for $k \in D_1$,

$$\frac{B(k)}{A(k)} = \frac{b(k)}{a(k)}, \quad k \in D_1.$$

Replacing k by $1/k$ in this equation, using equations (3.13), and solving the resulting equation for $B(k)/A(k)$ we find equation (1.10b).

It was shown in [1] that the solution of the basic RH problem is independent of T. Thus although the basic formula (1.10b) was derived under the assumption that $q(x,t) \to 0$ as $t \to \infty$, this formula is valid even without this assumption. **QED**

4 The Residue Conditions

We *assume* that $a(k)$ has n simple zeros $\{k_j\}_1^n$, n_1 of which are in D_1 and the remaining $n - n_1$ of which are in D_2.

We denote by $\dot{a}(k)$ the derivative of $a(k)$ with respect to k, and we denote by $[\mu]_1$ and $[\mu]_2$ the first and second columns of the 2×2 matrix μ.

Let $\{\lambda_j\}_1^\Lambda$ denote the zeros of $\Delta(k)$ for $k \in D_2$, where $\Delta(k)$ is defined by equation (1.12).

The following residue conditions are valid:

$$\begin{aligned} &\operatorname*{Res}_{k_j}[\mu(x,t,k)]_1 = \frac{e^{2i\theta(k_j)}}{\dot{a}(k_j)b(k_j)}[\mu(x,t,k_j)]_2, \quad j=1,\dots,n_1, \quad k_j \in D_1, \\ &\operatorname*{Res}_{\bar{k}_j}[\mu(x,t,k)]_2 = -\frac{e^{-2i\theta(\bar{k}_j)}}{\overline{\dot{a}(k_j)}\,\overline{b(k_j)}}[\mu(x,t,\bar{k}_j)]_1, \quad j=1,\dots,n_1, \quad \bar{k}_j \in D_4, \\ &\operatorname*{Res}_{\lambda_j}[\mu(x,t,k)]_1 = -\frac{\overline{N(\bar{\lambda}_j)}e^{2i\theta(\lambda_j)}}{\dot{a}(\lambda_j)\dot{\Delta}(\lambda_j)}[\mu(x,t,\lambda_j)]_2, \quad j=1,\dots,\Lambda, \quad \lambda_j \in D_2, \\ &\operatorname*{Res}_{\bar{\lambda}_j}[\mu(x,t,k)]_2 = -\frac{N(\lambda_j)e^{-2i\theta(\bar{\lambda}_j)}}{\overline{\dot{a}(\lambda_j)}\,\overline{\dot{\Delta}(\lambda_j)}}[\mu(x,t,\bar{\lambda}_j)]_1, \quad j=1,\dots,\Lambda, \quad \bar{\lambda}_j \in D_3, \end{aligned} \tag{4.1}$$

where $N(k)$ denotes the numerators of equations (1.10), and $a(k), b(k), \theta(x,t,k)$ are defined in Theorem 1.1.

In order to derive these equations we first recall that the matrix $\mu(x,t,k)$ appearing in the RH problem of Theorem 1.1 is constructed from three appropriate matrix solutions, $\{\mu_j(x,t,k)\}_1^3$, of the Lax pair (2.9). These matrices can be written in the column vector form

$$\mu_1 = (\mu_1^{(2)}, \mu_1^{(3)}), \quad \mu_2 = (\mu_2^{(1)}, \mu_2^{(4)}), \quad \mu_3 = (\mu_3^{(34)}, \mu_3^{(12)}), \tag{4.2}$$

where superscripts denote the domains that these column vectors are bounded and analytic ($\mu_1^{(2)}$ in D_2, $\mu_3^{(34)}$ in $D_3 \cup D_4$ etc.). It is shown in [1] that the matrix μ has the following form:

$$\mu_+ = \left(\frac{\mu_2^{(1)}}{a(k)}, \mu_3^{(12)}\right) \quad k \in D_1; \qquad \mu_- = \left(\frac{\mu_1^{(2)}}{d(k)}, \mu_3^{(12)}\right), \quad k \in D_2;$$

$$\mu_+ = \left(\mu_3^{(34)}, \frac{\mu_1^{(3)}}{\overline{d(\bar{k})}}\right) \quad k \in D_3; \qquad \mu_- = \left(\mu_3^{(34)}, \frac{\mu_2^{(4)}}{\overline{a(\bar{k})}}\right), \quad k \in D_4; \tag{4.3}$$

where

$$d(k) = a(k)\overline{A(\bar{k})} + b(k)\overline{B(\bar{k})}. \tag{4.4}$$

In order to derive equation (4.1a) we condsider the equation $\mu_- = \mu_+ J_4$, where J_4 is defined in Theorem 1.1 and μ_+, μ_- are given by the first and fourth equations in (4.3). The second column of this equation yields

$$\mu_3^{(12)} = a\mu_2^{(4)} + be^{-2i\theta}\mu_2^{(1)}.$$

Evaluating this equation at k_j we find

$$\mu_3^{(12)}(k_j) = b(k_j)e^{-2i\theta(k_j)}\mu_2^{(1)}(k_j),$$

where for simplicity of notation we suppress the (x,t) dependence of $\mu_3^{(12)}$ and $\mu_2^{(1)}$. Hence

$$\operatorname*{Res}_{k_j}[\mu]_1 = \frac{\mu_2^{(1)}(k_j)}{\dot{a}(k_j)} = \frac{e^{2i\theta(k_j)}}{\dot{a}(k_j)b(k_j)}\mu_3^{(12)}(k_j),$$

and since $\mu_3^{(12)} = [\mu]_2$, equation (4.1a) follows. The derivation of equation (4.1b) is similar.

In order to derive equation (4.1c) we consider the equation $\mu_- = \mu_+ J_1$, where J_1 is defined in Theorem 1.1 and μ_+, μ_- are given by the first and second equations in (4.3). The first column of this equation yields

$$\frac{\mu_1^{(2)}}{d(k)} = \frac{\mu_2^{(1)}}{a(k)} - \frac{\overline{N(\bar{k})}e^{2i\theta}}{a(k)[a(k)\overline{D(\bar{k})} + b(k)N(\bar{k})]}\mu_3^{(12)}. \tag{4.5}$$

Following arguments similar to those used in [3] it can be shown that the zeros of $d(k)$ in D_2 coincide with the zeros of $\Delta(k)$ in D_2, thus equation (4.5) yields

$$\mu_1^{(2)}(\lambda_j) = -\frac{\overline{N(\bar{\lambda}_j)}e^{2i\theta(\lambda_j)}}{a(\lambda_j)}\mu_3^{(12)}(\lambda_j).$$

Hence

$$\operatorname*{Res}_{\lambda_j}[\mu]_1 = \frac{\mu_1^{(2)}}{\dot{d}(\lambda_j)} = -\frac{\overline{N(\bar{\lambda}_j)}e^{2i\theta(\lambda_j)}}{a(\lambda_j)\dot{\Delta}(\lambda_j)}\mu_3^{(12)}(\lambda_j),$$

and since $\mu_3^{(12)} = [\mu]_2$, equation (4.1c) follows. The derivation of (4.1d) is similar.

5 Conclusion

It was shown in [1] and [3] that there exists a particular class of boundary conditions for which initial-boundary value problems on the half-line can be solved with the same level of efficiency as the classical initial value problem on the line. These "linearizable" boundary conditions were identified by the requirement that the eigenfunction $M(t,k)$, which satisfies the t-part of the associated Lax pair evaluated at $x = 0$ (as well as the condition that $M(0,k)$ equals the identity matrix), satisfies the "symmetry" relation

$$M(t,\nu(k)) = N(k)M(t,k)N(k)^{-1}; \tag{5.1}$$

in this equation the map $k \to \nu(k)$ is the map which leaves the dispersion relation of the linearized version of the given nonlinear PDE invariant (for the sG, $\nu(k) = 1/k$) and $N(k)$ is a non-singular matrix.

In this paper we have generalized equation (5.1) to the equation

$$M(t,\nu(k)) = F(t,k)M(t,k)F(0,k)^{-1}, \tag{5.2}$$

where $F(t,k)$ is a nonsingular matrix. Furthermore we have given an algorithmic way of constructing F by making use of the existence of different Lax pair formulations for the same nonlinear PDE. We expect that the systematic use of Bäcklund transformations will provide an approach to computing $F(t,k)$ for any linearizable boundary condition.

The main advantage of our method is *not* that it identifies linearizable boundary conditions, but that for such boundary conditions it expresses the solution $q(x,t)$ in terms of a simple Riemann-Hilbert (RH) problem. The basic features of this problem are similar with the basic features of the RH problem characterizing the solution of the classical initial-value problem, namely these two RH problems: (a) Have the *same* explicit (x,t) dependence. (b) Their jump matrices involve the functions $a(k)$ and $b(k)$ which are constructed from the initial conditions in a similar manner. Regarding differences between these two RH problems, we note that the RH problem associated with initial-boundary value problems has the novelty that it is formulated on a more complicated contour, and it also involves some additional jump functions which however can be *explicitly* written in terms of $a(k)$ and $b(k)$ (see equations (1.10)). The existence of a more complicated contour does *not* add any significant complexity to the analysis of the RH problem. Also in some cases it is possible to map this contour to the usual contour which is the real k-axis. This is actually the case for the sG and the nonlinear Schrödinger [7] (but *not* for the Korteweg-deVries and the modified Korteweg-deVries).

The simplicity of the basic RH problem has important implications for the analysis of the long time asymptotics. Indeed, in the case that the boundary conditions decay as $t \to \infty$, it is possible using the Deift-Zhou approach [8] to obtain a complete characterization of the solution as $t \to \infty$ and $x/t = O(1)$. The general structure of the asymptotics is given in [9], where it is shown that the solution is dominated by solitons. A detailed investigation of these solitons for the boundary condition (1.4b), as well as for the boundary condition

$$q(0,t) = \chi, \quad 0 < t < t_0; \quad q(0,t) = 0, \quad t_0 < t < \infty,$$

will be presented elsewhere. We also note that the simplicity of the basic RH problem makes it possible, using the Deift-Zhou-Venakides approach [10], to study the zero dispersion limit of initial-boundary value problems [7], [11].

Several authors have identified linearisable boundary conditions using the existence of symmetries and conservation laws, see for example [12]–[13]. The analysis of such boundary conditions using several formal RH problems was presented in [14]. The particular cases of either $\chi_1 = 0$ or $\chi_2 = 0$ are discussed in [15] using an extension of the problem from the half line to the infinite line (such an extension is *not* possible for PDEs with a third order derivative such as the KdV and the modified KdV equations). A discussion of the physical significance of the sG with the boundary condition (1.4b) as well as several approaches for the analysis of this problem can be found in [16]–[21]. In particular in [18] the question of the integrability of both the classical and quantum sine-Gordon theory involving $m^2 \sin(\beta q(x,t))/\beta$ with the boundary condition

$$\frac{\partial q(0,t)}{\partial x} + \frac{\partial V(q(0,t))}{\partial q} = 0, \tag{5.3}$$

was investigated. By demanding that a modification of the first non-trivial integral of motion of the usual theory remain conserved it was found that in general

$$V(q(0,t)) = \frac{Am}{\beta} \cos\left(\beta \frac{(q(0,t) - q_0)}{2}\right), \tag{5.4}$$

where A and q_0 are arbitrary real constants. In [18] the existence of an infinite number of integrals of motion for this system was assumed and the associated quantum field theory was studied. In [17] the question of integrability of the classical system with the boundary condition (5.3) was addressed. The results of [16] were used to prove that an infinite number of integrals of motion do exist, but only for certain $V(q)$. It was found that (5.4) is the most general boundary term compatible with the existence of infinitely many conserved quantities. This result thus agrees with [18]. The boundary condition (5.4) with $q_0 = 0$ or $q_0 = \pi/2$ has appeared in classical considerations of sine-Gordon theory. It was suggested in [18] that the scattering theory can depend on the extra parameter q_0; a similar question was investigated in [20] for real coupling affine Toda field theory.

Acknowledgments I am deeply grateful to E. Corrigan for suggesting this problem to me, and to A.R. Its for some important suggestions and in particular for showing me the relation between the two different Lax pair formulations of the sG equation.

References

[1] A.S. Fokas, Integrable nonlinear evolution equations on the half-line, Commun. Math. Phys. **230**, 1–39 (2002).

[2] A.S. Fokas, The Generalized Dirichlet to Neumann Map for Certain Nonlinear Evolution PDEs (preprint).

[3] A.S. Fokas, A.R. Its and L.-Y. Sung, The nonlinear Schrödinger equation on the half-line (preprint).

[4] L.D. Faddeev, L.A. Takhtajan and V.E. Zakharov, A complete description of the solutions of the sine-Gordon equation, DAN USSR, **219**, 1334–1337 (1974); L.D. Faddeev and L.A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, 1987.

[5] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Method for solving the sine-Gordon equation, Phys. Rev. Lett. **30**, 1262–1264 (1973).

[6] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. **21**, 467–490 (1968).

[7] A.S. Fokas and S. Kamvissis, The zero dispersion limit for integrable equations on the half-line with linearisable data, Applied and Abstract Analysis (in press).

[8] P.A. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems, Bull. Am. Math. Soc. **20**, 119–123 (1992); P.A. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, Ann. Math. **137**, 295–368 (1993).

[9] A.S. Fokas and A.R. Its, An initial-boundary value problem for the sine-Gordon equation in laboratory coordinates, Teoret. Matem. Fizika **92**, 387–403 (1992).

[10] P.A. Deift, S. Venakides and X. Zhou, New results in the small dispersion KdV by an extension of the method of steepest descent for Riemann-Hilbert problems, IMRN **6**, 285–299 (1997).

[11] S. Kamvissis, Semiclassical NLS on the half-line, J. Math. Phys. (to appear).

[12] I.T. Habibullin, Symmetry Approach in Boundary Value Problems, Nonlinear Math. Phys. **3**, 147–151 (1996).

[13] V.E. Adler, I.T. Habibullin and A.B. Shabat, Boundary value problem for the KdV equation on a half-line, Teoret. Matem. Fizika **110**, 98–113 (1997).

[14] I.T. Habibullin, KdV equation on a half-line with the zero boundary condition, Teoret. Matem. Fizika **119**, 397–404 (1999).

[15] V.O. Tarasov, The integrable initial-boundary value problem on the semiline: NLS and sG equations, Inv. Prob. **7**, 435–449 (1991).

[16] E.K. Sklyanin, Boundary conditions for integrable quantum systems, J. Phys. A, **21**, 2375–2389, (1988).

[17] A. MacIntyre, Integrable boundary conditions for classical sine-Gordon theory, J. Phys. A, **28**, 1089-1100, (1995).

[18] S. Ghoshal, A.B. Zamolodchikov, Boundary S matrix and boundary state in two-dimensional integrable quantum field theory, Int. J. Mod. Phys. A, **9**, 3841-3886 (1994), Erratum-ibid, A, **9**, 4353, (1994).

[19] E. Corrigan, P.E. Dorey, R.H. Rietdijk, R. Sasaki, Affine Toda field theory on a half-line, Phys. Lett. B **333**, 83–91 (1994).

[20] E. Corrigan, P.E. Dorey, R.H. Rietdijk, Aspects of affine Toda field theory on a half line, Prog. Theor. Phys. Suppl. **118**, 143–164, (1995).

[21] P. Bowcock, E. Corrigan, P.E. Dorey, R.H. Rietdijk, Classically integrable boundary conditions for affine Toda field theories, Nucl. Phys. B, **445**, 469–500, (1995).